MA/STAT 170 Lecture 1

Simple interest: only pay interest on the principal - does not compound - eg child

$A(t) = P + itP = (1+it)P$ $\quad t$ = time

i = interest rate e.g. .05

Compound interest eg. 1% / year

General formula: P = principal

1 year $\quad A(1) = P + iP = (1+i)P$

2 years $\quad A(2) = (1+i)P + i(1+i)P$
$= (1+i)(1+i)P = (1+i)^2 P$

t years $\quad A(t) = (1+i)^t P$

Example On Jan. 1 1980 I deposited \$10,000 in an account earning 5% interest compounded annually.

How much did I have on Jan 1. 2010?

Solución My money was on account a full 30 years. Hence I have

$(1+.05)^{30} \times 10,000 = 43,219.42$

Remark How to count days

Example On Jan. 1 2008 I deposits \$ 1000 in an account earning 7.5% compound interest. I make the following deposits and withdraws:

Jan. 1 2008 + \$ 1000

Jan. 1 2009 − \$ 500

Jan 1 2010 + \$ 1500

What will my balane be on Jan 1. 2012?

Solución Hard way:

Balance: Jan 1. 2009

$1000(1.075) - 500 = 575$

Jan 1 2010

$$575 \times (1.075) + 1500 = \underline{2164.48}$$

Jan 1 2012

$$2164.48 \times (1.075)^2 = 2501.34$$

Easy way

$$1000(1.075)^4 - 500(1.075)^3 + 1500(1.075)^2$$

$$= 2501.34$$

We can always treat deposites and withdrawals separately

Example I borrow \$550 at 4% interest. I make the following end of year payments

Year 1 : \$100

Year 2: \$300

Year 3: $-\$50$ (Borrow \$50 more)

Year 4: Final payment

What was the final payment?

Solution We treat each payment and loan separately

Including interest at the end of 4 years I owe

$$\underbrace{(1.04)^4\, 550}_{4 \text{ years}} + \underbrace{(1.04)\, 50}_{1 \text{ year}} = 695.42$$

My payments result in an interest saving reducing the amount I owe by

$$(1.04)^4\, 100 + (1.04)^3 \cdot 300 = 436.97$$

Thus, in the end, I owe

Final paymen

$$\underset{\text{No payments}}{695.42} - \underset{\text{payments + interest saving}}{436.97} = 258.45$$

Recall the second Example above:

Example On Jan. 1 1980 I deposited

$10,000 in an account earning

5% interest compounded annually.

Suppose I withdraw my money

after 6 mo. How much do I get?

Some Possibilities

Compound Interest:

$(1.05)^{\frac{1}{2}} \cdot 10{,}000 = 10{,}246$ (Not typical)

Simple Interest

$\left(1 + \frac{.05}{2}\right) 10{,}000 = 10{,}250$

I pay a penalty

over short intervals of time $(0 \le t \le 1)$

simple interest & compound interest

yeild almost the same amounts

Mathematically

$$(1+i)^t \approx 1 + it \quad , \ 0 \le t \le 1 \quad , \ 0 < i < 1$$

Lecture 2

Exercise (19) p. 13

(19) One bank is paying 4.8% compounded monthly. Another bank is paying 5% annual effective. Which is paying more?

Monthly compounding at rate i means that every month the account grows by a factor of $(1+\frac{i}{12})$. Thus, in 1 year it will grow by $(1+\frac{i}{12})^{12}$. Thus, in this exercise, P dollars in the first bank grows to

$$\left(1+\frac{.048}{12}\right)^{12} P = (1.04907) P$$

Hence their annual effective rate is 4.907 % which still is not as good as 5%

Definition The symbol $i^{(m)}$ means that the interest rate i is being compounded n times a year. Thus in exercise (19) we could say that Bank 1 is paying $4.8^{(12)}$. 4.8 is called the "nominal rate." The annual effective rate is 4.907

(21) What is the present value of a payment of $12,000 to made at the end of 6 years if the interest rate is 7% effective?

This is the same as asking "How much do I need to invest today to obtain $12,000 six years from now if I can invest money at 7% interest. The solution is simple

$$(1.07)^6 P = 12000$$

$$P = (1.07)^{-6} \cdot 12000 = 7996.11$$

We say that "At 7% interest"

the present value of \$ 12,000

six years from now is \$7,996.11.

If we can earn 7% interest a

promise to pay \$12,000 six years

from now is worth only \$7,996.11

today.

Formula $PV = (1+i)^{-n} FV$

(22) What is the value in eight years (i.e. the future value) of a payment now of \$45,000 if the interest rate is 4.5% effective?

Future value is the reverse of

present value. This is the same

as asking "If 45,000 is invested

at 4.5% intrest now, how much will

you have in 8 years. The answer is

$(1.045)^8 \cdot 45000 = 63{,}994.53$

At 4.5% interest, \$45,000 today is worth \$63,994.53 8 years from now.

$$FV = (1+i)^n PV.$$

(47) I buy a piano from Cheapside Music company on March 1, 1998. I pay \$1,000 immediately, then 3 more payments of \$1,000 on March 1 for each of the next 3 years. Finally, on March 1, 2002, I pay \$10,000. What was this deal worth to Cheapside on March 1, 1998, assuming that they can invest funds at 4% interest–i.e. what was the present value of all of my payments on March 1, 1998?

This is a very important problem. Cheapside needs to know how much money they are making from selling the piano. They receive 4 payments of \$1000 and a final payment of \$10,000 suggesting that they make \$14,000 - but this is incorrect. \$10,000 paid 4 years now is not worth \$10,000 now. Similarly, each of the \$1,000 payments, except the first payment, which was paid immediately, was not actually worth \$1,000 to Cheapside. Since the payments were all made at differing times, we value each separately and sum the results:

$$PV = 1000 + (1.04)^{-1} \cdot 1000 + (1.04)^{-2} \cdot 1000 + (1.04)^{-3} \cdot 1000 + (1.04)^{-4} \cdot 10{,}000$$

$$= 12{,}323.13$$

This represents how much Cheapside is actuall earning on the deal.

Relate to Actuarial Science

Annuities

A crucial formula:

$$1+x+x^2+\dots+x^n = \frac{x^{n+1}-1}{x-1} \quad \text{(geometric series)}$$

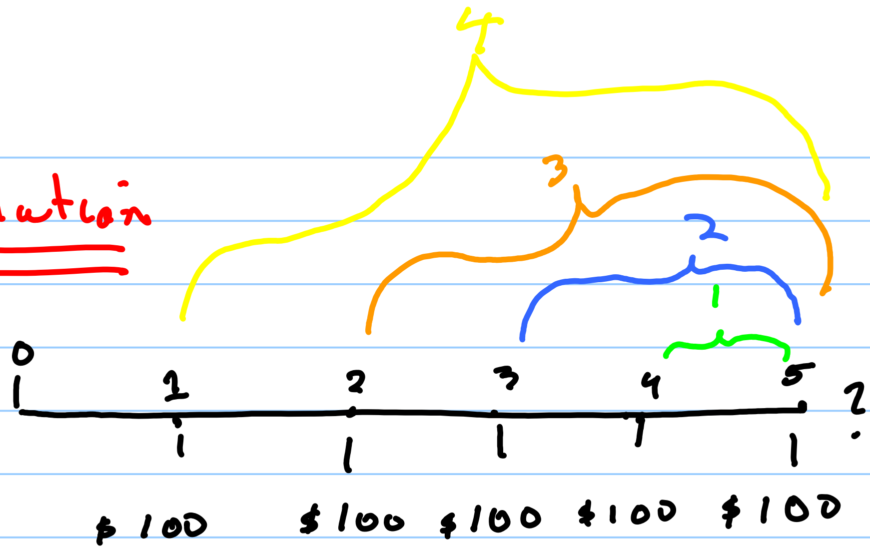
Proof

$$(x-1)(1+x+x^2+\dots+x^n) = -1\cdot(1+x+x^2+\dots+x^n)$$
$$+x\cdot(1+x+x^2+\dots+x^n)$$
$$= -1 - \cancel{x} - \cancel{x^2} \dots - \cancel{x^n}$$
$$+ \cancel{x} + \cancel{x^2} + \dots + \cancel{x^n} + x^{n+1}$$
$$= x^{n+1}-1$$

$$1+x+x^2+\dots+x^n = \frac{x^{n+1}-1}{x-1}$$

Example I deposite \$100 at the end of the year for 5 years into an account earning 3% /year. What is my total accumulation

Solution



Answer:

$$100\cdot(1.03)^4 + 100(1.03)^3 + 100\cdot(1.03)^2 + (1.03)\cdot 100 + 100.$$

end of last year - no interest

$$= 100\left(1 + (1.03) + (1.03)^2 + (1.03)^3 + (1.03)^4\right)$$

Let $x = 1.03$, $n = 4$

$$1 + x + x^2 + \cdots + x^n = \frac{x^{n+1} - 1}{x - 1}$$

5 years

$$1 + (1.03) + (1.03)^2 + (1.03)^3 + (1.03)^4 = \frac{(1.03)^5 - 1}{1.03 - 1}$$

$$= \frac{(1.03)^5 - 1}{.03}$$

$$= 5.3091$$

Our answer \$530.91

General Formula

n deposites of D made at the end of the year will accumulate to

$$\left(\frac{(1+i)^n-1}{i}\right)D$$

Example I deposite \$100 at the beginning of the year for 5 years into an account earning 3% /year. What is my total accumulation?

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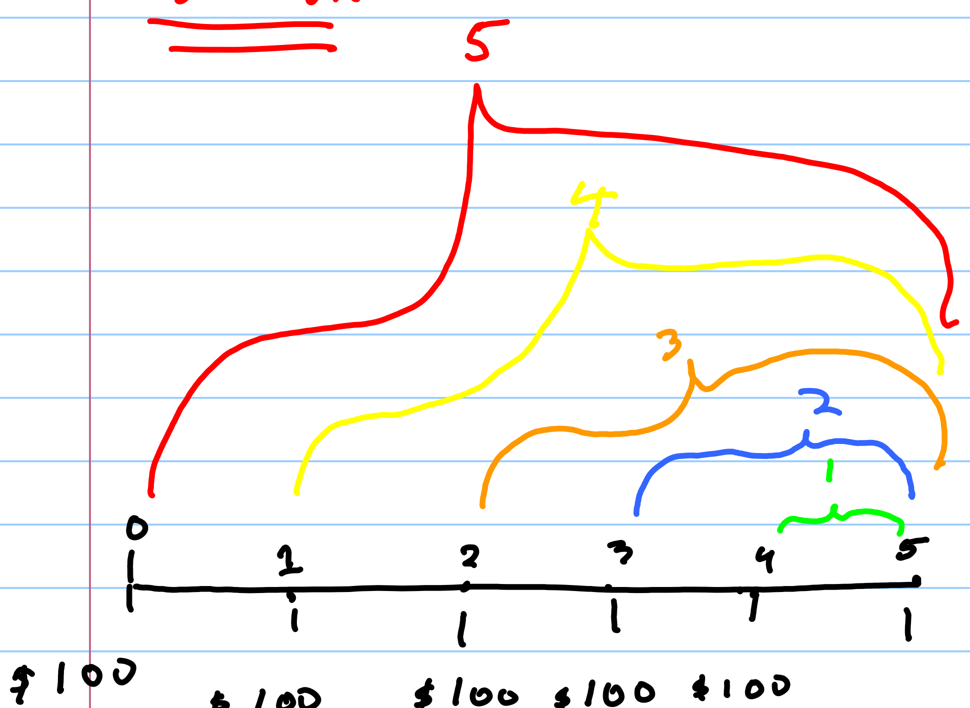

we still make 5 deposits, but each deposit earns one more year of interest.

Hence our answer is

$$(1.03)\cdot 503.91 = 519.03$$

General Formula

n deposites of D made at the beginning of the year will accumulate to

$$(1+i)\left(\frac{(1+i)^n-1}{i}\right)D$$